

A Characterization of the Heat Kernel Coefficients

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Abstract

We consider the asymptotic expansion of the heat kernel of a generalized Laplacian for $t \rightarrow 0^+$ and characterize the coefficients a_k , $k \geq 0$, of this expansion by a natural intertwining property. In particular we will give a closed formula for the infinite order jet of these coefficients on the diagonal in terms of the local expressions of the powers of the given generalized Laplacian in normal coordinates.

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1 Introduction

A generalized Laplacian Δ acting on sections of a vector bundle \mathcal{E} over a Riemannian manifold M is a second order elliptic differential operator with scalar symbol ([BGV]). Its spectrum is interesting for various geometrical questions and the spectra of the more prominent generalized Laplacians are of fundamental importance in global analysis. The short distance behaviour of the spectrum of a generalized Laplacian can be studied using analytic properties of its heat operator K_t^Δ , which solves the heat equation

$$\frac{\partial}{\partial t} (K_t^\Delta \psi) = -\Delta \psi$$

associated to Δ for all sections ψ of \mathcal{E} . The general existence result of Minakshisundaram–Pleijel ([G],[BGV]) shows that K_t^Δ is an integral operator for all $t > 0$ with integral kernel $k_t^\Delta(x, y)$, which is a smooth section of the bundle with fiber $\text{Hom}(\mathcal{E}_x, \mathcal{E}_y)$ over the point (x, y) of the product $M \times M$. Except for a very limited set of examples the heat kernel is not known explicitly, perhaps the most important exception is the heat kernel of the flat Laplacian Δ acting on sections of a trivial vector bundle $V \times E$ over a euclidian vector space V of dimension n :

$$k_t^\Delta(x, y) = \frac{1}{\sqrt{4\pi t}^n} e^{-\frac{|x-y|^2}{4t}} \text{id}_E$$

As can be seen from this example the heat kernel $k_t^\Delta(x, y)$ peaks on the diagonal $\{x = y\}$ of $M \times M$ as $t \rightarrow 0^+$ and it turns out that the rate of its divergence on the diagonal carries a

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tremendous amount of information about the geometry of the underlying manifold M . More precisely the heat kernel has an asymptotic expansion along the diagonal as $t \rightarrow 0^+$ of the form

$$k_t^\Delta(x, y) \underset{t \rightarrow 0^+}{\sim} \frac{1}{\sqrt{4\pi t}} e^{-\frac{\text{dist}^2(x, y)}{4t}} \sum_{k \geq 0} t^k \hat{a}_k(x, y) \quad (1)$$

where $\text{dist}^2(x, y)$ is the square of the geodesic distance from x to y and the $\hat{a}_k(x, y)$ are again sections of the bundle $\text{Hom}(\mathcal{E}_x, \mathcal{E}_y)$. In a sense these coefficients \hat{a}_k , $k \geq 0$, describe how much the heat kernel of the operator Δ differs in divergence from the flat Laplacian Δ in various directions and this difference does not only depend on the coefficients of Δ , but involves the geometry of M in a crucial way.

As a matter of convenience we will not use the coefficients $\hat{a}_k(x, y)$ directly but will multiply them $a_k(x, y) := j(x, y)\hat{a}_k(x, y)$ by the jacobian determinant $j(x, y)$ of the differential of the exponential map in y at x , which is a well defined smooth function in a neighborhood of the diagonal. Off this neighborhood we may extend $j(x, y)$ arbitrarily to a smooth function on $M \times M$, because only the germs of the coefficients $\hat{a}_k(x, y)$ along the diagonal are characterized by the asymptotic expansion (1) of k_t^Δ above. The main theorem of this article exhibits a remarkable intertwining property of these modified coefficients a_k , $k \geq 0$:

Theorem 2.1 (The Intertwining Property)

Let Δ be a generalized Laplacian acting on sections ψ of a vector bundle \mathcal{E} over M . The coefficients $a_k : \mathcal{E}_x \rightarrow \mathcal{E}_y$ of the asymptotic expansion of the heat kernel of Δ satisfy

$$\frac{(-1)^\mu}{\mu!} [\Delta^\mu \psi](y) = \sum_{\nu=0}^{\mu} \frac{(-1)^\nu}{\nu!} [\Delta^\nu (a_{\mu-\nu}(\exp_y \cdot, y) \psi(\exp_y \cdot))](0)$$

where $a_{\mu-\nu}(\exp_y \cdot, y) \psi(\exp_y \cdot)$ is considered as a section of the trivial vector bundle $T_y M \times \mathcal{E}_y$ over $T_y M$ and Δ is the flat Laplacian acting on sections of this trivial bundle.

The rest of the article is devoted to corollaries of Theorem 2.1. In particular we prove the striking fact that the intertwining property alone characterizes the infinite order jet of all coefficients $a_k(\cdot, y)$ in y and on the way we will give a different proof of Polterovich's inversion formula ([P1]) for the endomorphisms $a_k(y, y)$, $k \geq 0$, of the fiber \mathcal{E}_y in a somewhat stronger formulation:

Theorem 2.4 (Polterovich's Inversion Formula)

Let Δ be a generalized Laplacian acting on sections of a vector bundle \mathcal{E} over M and let $\text{dist}^2(\cdot, y)$ be the square of the geodesic distance to $y \in M$. For all $r \geq k \geq 0$ and all sections ψ of \mathcal{E} the endomorphism $a_k(y, y) = \hat{a}_k(y, y)$ of the fiber \mathcal{E}_y is given by the formula:

$$[a_k \psi](y) = \sum_{l=0}^r (-\tfrac{1}{4})^l \binom{r + \frac{n}{2}}{r-l} \left[\frac{(-1)^{k+l}}{(k+l)!} \Delta^{k+l} \left(\frac{1}{l!} \text{dist}^{2l}(\cdot, y) \psi \right) \right](y).$$

The final result is Theorem 2.7, which provides an explicit formula for the infinite order jets of the coefficients $a_k(\cdot, y)$ in y in terms of the local expressions of the powers of the operator Δ in normal coordinates on M and some smooth trivialization of \mathcal{E} .

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2 The Intertwining Property

It is well known that the coefficients $\hat{a}_k, k \geq 0$, of the asymptotic expansion of the heat kernel of Δ are local expressions in the curvature of M and the coefficients of Δ ([G]). Consequently we will assume throughout this section that Δ is defined on sections of a trivializable vector bundle \mathcal{E} over a starshaped neighborhood of the origin in some euclidean vector space V of dimension n . Moreover we will assume that the identity map $V \longrightarrow V$ is the exponential map in the origin with respect to the Riemannian metric defined by the symbol of Δ , a fortiori the symbol metric must agree with the scalar product of V in 0. The general case is easily reduced to this local model by taking $V = T_y M$ to be the tangent space of the underlying manifold at the point y under consideration and choosing the neighborhood of the origin and the trivializable vector bundle \mathcal{E} accordingly.

The preceding assumptions are crucial to the entire argument but single out the origin as a special point. Consequently all our statements below refer to the fixed target point $y = 0$. In particular the coefficients $\hat{a}_k(x) := \hat{a}_k(x, 0)$ will be homomorphisms $\hat{a}_k(x) : \mathcal{E}_x \longrightarrow E$ from the fiber of \mathcal{E} over x to the fiber E of \mathcal{E} over the origin. Similarly we will write $j(x) := j(x, 0)$ for the Jacobian determinant of the exponential map in the origin at the point $x \in V$, which is the identity map $V \longrightarrow V$ by assumption. In other words the Jacobian determinant is characterized by $d\text{vol}_g = j(x)dx$ and is the square root $j(x) := \sqrt{\det g(x)}$ of the determinant of the symbol metric with respect to the flat scalar product. Now by the remarks following ([BGV, Theorem 2.30]) we have an asymptotic expansion for the heat operator

$$[K_t^\Delta \psi](0) \underset{t \rightarrow 0^+}{\sim} \sum_{\mu \geq 0} \frac{(-t)^\mu}{\mu!} [\Delta^\mu \psi](0)$$

which complements the asymptotic expansion (1) of its integral kernel k_t^Δ . Using both asymptotic expansions we find for any smooth section ψ of \mathcal{E} with compact support in a sufficiently small neighborhood of the origin

$$\begin{aligned} \int_V k_t^\Delta(x) \psi(x) d\text{vol}_g &\underset{t \rightarrow 0^+}{\sim} \sum_{\mu \geq 0} \frac{(-t)^\mu}{\mu!} [\Delta^\mu \psi](0) \\ &\underset{t \rightarrow 0^+}{\sim} \int_V \frac{1}{\sqrt{4\pi t}^n} e^{-\frac{|x|^2}{4t}} \sum_{\mu \geq 0} t^\mu \hat{a}_\mu(x) \psi(x) d\text{vol}_g \\ &\underset{t \rightarrow 0^+}{\sim} \sum_{\mu \geq 0} t^\mu \int_V \frac{1}{\sqrt{4\pi t}^n} e^{-\frac{|x|^2}{4t}} (j(x) \hat{a}_\mu(x) \psi(x)) dx \\ &\underset{t \rightarrow 0^+}{\sim} \sum_{\mu \geq 0} t^\mu \sum_{\nu \geq 0} \frac{(-t)^\nu}{\nu!} [\Delta^\nu (a(x) \psi(x))](0), \end{aligned}$$

where $a(x) := j(x)\hat{a}_k(x)$ and Δ is the flat Laplacian acting on sections of the trivial E -bundle $V \times E$ over V . Note that the asymptotic expansion of the heat kernel k_t^Δ is locally uniform in x according to ([BGV, Theorem 2.30]) and hence we may integrate it over the compact support of ψ to obtain an asymptotic expansion of the integral. Existence of asymptotic expansions implies uniqueness and so we need only sort out the different powers of t to prove the intertwining property of the heat kernel coefficients in the form:

Theorem 2.1 (Intertwining Property of the Heat Kernel Coefficients)

Let Δ be a generalized Laplacian acting on sections of a trivializable vector bundle \mathcal{E} over V . The coefficients a_k , $k \geq 0$, of the asymptotic expansion of the heat kernel of Δ are sections of the bundle $\text{Hom}(\mathcal{E}, V \times E)$ and so $a_k\psi$ is a well defined section of $V \times E$ for any section ψ of \mathcal{E} . The coefficients a_k intertwine the powers of Δ and Δ :

$$\frac{(-1)^\mu}{\mu!} [\Delta^\mu \psi](0) = \sum_{\nu=0}^{\mu} \frac{(-1)^\nu}{\nu!} [\Delta^\nu (a_{\mu-\nu} \psi)](0). \quad (2)$$

A direct consequence of Theorem 2.1 is that a generalized Laplacian Δ acting on sections of a vector bundle \mathcal{E} remembers to vanish on sections ψ of \mathcal{E} , which are harmonic in some smooth trivialization of \mathcal{E} , in the following sense:

Remark 2.2 (Excess Vanishing for Harmonic Sections)

Consider a section ψ of \mathcal{E} , which is a homogeneous, harmonic polynomial $\Phi\psi \in \text{Sym}^d V^* \otimes E$ of degree d with $\Delta(\Phi\psi) = 0$ in some smooth trivialization Φ of \mathcal{E} . If Δ is a generalized Laplacian acting on sections of \mathcal{E} , then $[\Delta^\mu \psi](0) = 0$ vanishes for all $\mu < d$.

Philosophically the intertwining property (2) provides a precise geometric interpretation for the coefficients a_k of the heat kernel expansion in terms of Δ and the flat model operator Δ for generalized Laplacians. Similar considerations should apply to other model operators arising e. g. in Heisenberg calculus or in other parabolic calculi. A convenient reformulation of this property (2) can be given with the help of the formal power series $e^{-z\Delta}$ and $e^{-z\Delta}$ of differential operators on \mathcal{E} and $V \times E$ respectively and the generating series

$$a(z) := \sum_{\mu \geq 0} a_\mu z^\mu$$

for the heat kernel coefficients a_k , $k \geq 0$. Namely the intertwining property (2) is just another way to write down the equality $[e^{-z\Delta}\psi](0) = [e^{-z\Delta}(a(z)\psi)](0)$ of formal power series in z for all sections ψ , which in turn is equivalent to the commutativity of the pentagram

$$\begin{array}{ccc} \text{Jet}_0^\infty \mathcal{E} & \xrightarrow{a(z)} & \text{Jet}_0^\infty E[[z]] \\ \downarrow e^{-z\Delta} & & \downarrow e^{-z\Delta} \\ \text{Jet}_0^\infty \mathcal{E}[[z]] & & \text{Jet}_0^\infty E[[z]] \\ \searrow \text{ev} & & \swarrow \text{ev} \\ & E[[z]] & \end{array}$$

where ev is the evaluation at the origin. Only the infinite order jet of the sections a_k , $k \geq 0$, of $\Gamma(V, \text{Hom}(\mathcal{E}, V \times E))$ in the origin can ever be sensed by jet theory of course, and so the commutativity of this square does only depend on the infinite order jet of the generating series $a(z)$. Strikingly however the commutativity of the pentagon characterizes the infinite order jet of $a(z)$ completely. In fact the unique jet solution of the intertwining property thought of as a set of equations for the unknowns a_k , $k \geq 0$, is given in Theorem 2.7 below.

At this point we digress a little bit on a very interesting property of the flat Laplacian Δ acting on functions $C^\infty V$ on a euclidian vector space V . More general we can consider the flat Laplacian Δ acting on sections $C^\infty(V, E)$ of the trivial E -bundle $V \times E$ over V , but the auxiliary vector space E never enters into the argument directly and so we will stick to the case $E = \mathbb{R}$. Along with the flat Laplacian Δ comes the operator $|x|^2$ of multiplication by the square of the distance to the origin, in orthogonal coordinates $\{x_\mu\}$ on V these two operators can be written:

$$\Delta := - \sum_{\mu} \frac{\partial^2}{\partial x_\mu^2} \quad |x|^2 := \sum_{\mu} x_\mu^2.$$

By restriction they act on the subspace $\text{Sym } V^* \subset C^\infty V$ of polynomials on V and their commutator $[\Delta, |x|^2] = (-4)(N + \frac{n}{2})$ as operators on $\text{Sym } V^*$ involves the number operator $N : \text{Sym } V^* \rightarrow \text{Sym } V^*$, which multiplies $\text{Sym}^r V^*$ by r , shifted by half the dimension n of V . In other words $X := \frac{1}{2}|x|^2$ and $Y := \frac{1}{2}\Delta$ close with $H := N + \frac{n}{2}$ to an \mathfrak{sl}_2 -algebra of operators on $\text{Sym } V^*$. Iterated commutators of X and Y in \mathfrak{sl}_2 -representations can in general be written down using factorial polynomials $[x]_r := x(x-1)\dots(x-r+1)$ or binomial coefficients $\binom{x}{r} := \frac{1}{r!}[x]_r$ for integral $r \geq 0$. In particular the standard relation $Y^r X^r \psi = r! [-\lambda]_r \psi$ for all ψ with $Y \psi = 0$ and $H \psi = \lambda \psi$ implies for the constant function $\psi := 1 \in \text{Sym}^0 V^*$ the classical formula:

$$[\Delta^r |x|^{2r}](0) = 4^r r! [-\frac{n}{2}]_r. \quad (3)$$

Slightly more useful for our purposes is the following derived identity:

Lemma 2.3 *For all smooth functions $\psi \in C^\infty V$ and all $k, l \geq 0$:*

$$\left[\frac{(-1)^{k+l}}{(k+l)!} \Delta^{k+l} \left(\frac{1}{k!} |x|^{2k} \psi \right) \right] (0) = (-4)^k \binom{-\frac{n}{2}-l}{k} \left[\frac{(-1)^l}{l!} \Delta^l \psi \right] (0).$$

Proof: Only a finite number of partial derivatives of ψ in the origin 0 are actually involved in this identity and hence we may assume that ψ is a polynomial without loss of generality. Moreover only the homogeneous component of ψ of degree $2l$ contributes to left and right hand side, which are both evidently $\mathbf{SO } V$ -invariant linear functionals in $\psi \in \text{Sym}^{2l} V^*$. However there is but one $\mathbf{SO } V$ -invariant linear functional on $\text{Sym}^{2l} V^*$ up to scale, so that it is sufficient to check the identity in question, which can be rewritten as

$$[\Delta^{k+l} (|x|^{2k} \psi)](0) = \frac{4^{k+l} (k+l)! [-\frac{n}{2}]_{k+l}}{4^l l! [-\frac{n}{2}]_l} [\Delta^l \psi](0),$$

for the single polynomial $\psi := |x|^{2l}$, for which it is true by the classical formula (3). \square

Returning to the general case of a Laplacian Δ acting on sections of a vector bundle \mathcal{E} over V we recall that the generating series $a(z)$ for the coefficients in the asymptotic expansion of the heat kernel intertwines the formal power series $e^{-z\Delta}$ and $e^{-z\Delta}$ of differential operators. Using this intertwining property (2) together with Lemma 2.3 we calculate:

$$\begin{aligned} \left[\frac{(-1)^{k+l}}{(k+l)!} \Delta^{k+l} \left(\frac{1}{l!} |x|^{2l} \psi \right) \right] (0) &= \sum_{\mu=0}^{k+l} \left[\frac{(-1)^\mu}{\mu!} \Delta^\mu \left(\frac{1}{l!} |x|^{2l} a_{k+l-\mu} \psi \right) \right] (0) \\ &= \sum_{\mu=0}^k \left[\frac{(-1)^{l+\mu}}{(l+\mu)!} \Delta^{l+\mu} \left(\frac{1}{l!} |x|^{2l} a_{k-\mu} \psi \right) \right] (0) \\ &= (-4)^l \sum_{\mu=0}^k \binom{-\frac{n}{2}-\mu}{l} \left[\frac{(-1)^\mu}{\mu!} \Delta^\mu (a_{k-\mu} \psi) \right] (0). \end{aligned}$$

Combining this equation with the binomial inversion formula

$$\sum_{l=0}^r \binom{r+\frac{n}{2}}{r-l} \binom{-\frac{n}{2}-\mu}{l} = \binom{r-\mu}{r} = \delta_{\mu,0}$$

valid as soon as $r \geq \mu \geq 0$ we eventually arrive at the following inversion formula for the heat kernel coefficients a_k , $k \geq 0$, of a generalized Laplacian Δ :

Theorem 2.4 (Polterovich's Inversion Formula ([P1],[P2]))

Let Δ be a generalized Laplacian acting on sections of a vector bundle \mathcal{E} over V and let $|x|$ be the radial distance from the origin. For any section $\psi \in \Gamma(V, \mathcal{E})$ and all $r \geq k \geq 0$ we can compute the action of the endomorphism $a_k(0)$ of the fiber E of \mathcal{E} over the origin on ψ by means of an explicit inversion formula:

$$[a_k \psi](0) = \sum_{l=0}^r \left(-\frac{1}{4}\right)^l \binom{r+\frac{n}{2}}{r-l} \left[\frac{(-1)^{k+l}}{(k+l)!} \Delta^{k+l} \left(\frac{1}{l!} |x|^{2l} \psi \right) \right] (0).$$

Our normalization of the coefficients a_k , $k \geq 0$, drops the factor $(4\pi)^{-\frac{n}{2}}$ arising from the value of the euclidian heat kernel for the flat Laplacian Δ on V at the origin to have the intertwining property (2) in as simple a form as possible. In the original formulation of Polterovich this factor is part of a_k and of course in every conceivable application this factor has to be reinserted by hand.

Example 2.5 (The Local Index Theorem for the Gauß–Bonnet Operator)

Choose a smooth Riemannian metric g on V satisfying our standing assumptions and a vector bundle \mathcal{E} over V with a connection ∇ . For a generalized Laplacian on \mathcal{E} of the form $\Delta_{\hbar} = \nabla^ \nabla + \hbar F$ for some smooth endomorphism F of \mathcal{E} we evidently have*

$$\begin{aligned} \frac{(-1)^{k+l}}{(k+l)!} (\nabla^* \nabla + \hbar F)^{k+l} &= \frac{(-\hbar)^k}{k!} F^k \frac{(-1)^l}{l!} \Delta^l + \text{terms of lower order in } \hbar \\ &\quad + \text{differential operators of lower order} \end{aligned}$$

for all $k, l \geq 0$. Plugging this into Polterovich's Inversion formula we verify immediately

$$a_k(0) = \frac{(-\hbar)^k}{k!} F^k + \text{terms of lower order in } \hbar$$

for all $k \geq 0$, which may have been anticipated from Theorem 2.1. Remarkably this simple calculation together with a few general considerations concerning the order of the coefficients a_k is already sufficient to prove the local index theorem for the Gauß–Bonnet operator!

In conclusion the intertwining property (2) alone is sufficient to determine the value of the coefficients a_k , $k \geq 0$, of the heat kernel expansion at 0. Before we proceed to show that in fact the jets of infinite order $\text{jet}_0^\infty a_k$, $k \geq 0$, are determined by (2) we want to make a few general remarks concerning our guiding philosophy in the calculations to come. We want to avoid formulas involving compositions of differential operators, because such formulas are hardly if ever of any use in explicit calculations. In favourable situations it may still be possible to calculate the values $\text{ev } \mathcal{D}^r$, $r \geq 0$, of powers of a differential operator \mathcal{D} in the origin without knowing the partial derivatives of their coefficients. Note that the usual arguments of symbol calculus become meaningless if we have no control over the partial derivatives of the differential operators in a singular point like the origin.

In general the value of a differential operator \mathcal{D} acting on sections of \mathcal{E} at the origin will be an element of $\text{Hom}(\text{Jet}_0^\infty \mathcal{E}, E)$. For the trivial E -bundle $V \times E$ however we may identify $\text{Jet}_0^\infty(V \times E)$ with the formal power series completion of $\text{Sym } V^* \otimes E$ and hence $\text{Hom}(\text{Jet}_0^\infty(V \times E), E)$ with $\text{Sym } V \otimes \text{End } E$ in the usual way. The scalar product of the euclidian vector space V extends to a scalar product on $\text{Sym } V^*$ defined via Gram's permanent and characterized by $\langle e^\alpha, e^\beta \rangle_{\text{Sym } V^*} = e^{\langle \alpha, \beta \rangle}$ for all $\alpha, \beta \in V^*$ with a slight abuse of notation. Alternatively we may choose orthonormal coordinates $\{x_\mu\}$ on V and write down the scalar product directly

$$\begin{aligned} \langle \psi, \tilde{\psi} \rangle_{\text{Sym } V^*} &:= \sum_{r \geq 0} \frac{1}{r!} \sum_{\mu_1, \dots, \mu_r} \left[\frac{\partial^r}{\partial x_{\mu_1} \dots \partial x_{\mu_r}} \psi \right] (0) \left[\frac{\partial^r}{\partial x_{\mu_1} \dots \partial x_{\mu_r}} \tilde{\psi} \right] (0) \\ &= ((\text{ev} \otimes \text{ev}) \circ e^{\langle \nabla, \nabla \rangle}) (\psi \otimes \tilde{\psi}) \end{aligned}$$

where $\langle \nabla, \nabla \rangle$ is the bidifferential operator $\psi \otimes \tilde{\psi} \mapsto \sum_\mu \frac{\partial}{\partial x_\mu} \psi \otimes \frac{\partial}{\partial x_\mu} \tilde{\psi}$ and ev is the evaluation at 0 as before. Written in this form it is clear that the scalar product extends to the formal power series completion of $\text{Sym } V^*$ or even to smooth functions provided that the defining sum converges. Moreover the musical isomorphism $\sharp : \text{Sym } V \rightarrow \text{Sym } V^*$ of this scalar product is the natural extension of the musical isomorphism of V and extends to the formal power series completion of $\text{Sym } V$, too. The value of a differential operator \mathcal{D} acting on sections of $V \times E$ has an image $(\text{ev } \mathcal{D})^\sharp$ under this musical isomorphism, which is the unique polynomial on V satisfying $[\mathcal{D}\psi](0) = \langle (\text{ev } \mathcal{D})^\sharp, \psi \rangle_{\text{Sym } V^*}$ for all smooth sections ψ of $V \times E$.

The scalar product on $\text{Sym } V^*$ will enter the formulas through the operator $\langle \nabla, \nabla \rangle$ introduced above, which in turn makes its appearance via Green's identity for the flat Laplacian Δ and the multiplication map m

$$(\Delta \circ m)(\psi \otimes \tilde{\psi}) = (m \circ (\Delta \otimes \text{id} - 2\langle \nabla, \nabla \rangle + \text{id} \otimes \Delta))(\psi \otimes \tilde{\psi})$$

which features three commuting operators $\Delta \otimes \text{id}$, $\text{id} \otimes \Delta$ and $\langle \nabla, \nabla \rangle$. Hence we are free to put these operators in arbitrary order upon exponentiation. For the ensuing calculations we need to choose a trivialization Φ of \mathcal{E} , which we think of as a family of homomorphisms $\Phi(x) : \mathcal{E}_x \longrightarrow E$ with $\Phi(0) = \text{id}_E$, so that $\Phi\psi$ is a smooth section of $V \times E$ for every section ψ of \mathcal{E} . With this in mind we find

$$\begin{aligned} [e^{-z\Delta} (a(z)\psi)](0) &= (\text{ev} \circ e^{-z\Delta} \circ m)(a(z)\Phi^{-1} \otimes \Phi\psi) \\ &= ((\text{ev} \otimes \text{ev}) \circ e^{2z\langle \nabla, \nabla \rangle})(e^{-z\Delta}(a(z)\Phi^{-1}) \otimes e^{-z\Delta}(\Phi\psi)) \\ &= \langle (2z)^N (e^{-z\Delta}(a(z)\Phi^{-1})), (e^{-z\Delta}(\Phi\psi)) \rangle_{\text{Sym } V^*} \\ &= \langle e^{z|x|^2} (2z)^N (e^{-z\Delta}(a(z)\Phi^{-1})), \Phi\psi \rangle_{\text{Sym } V^*} \end{aligned}$$

while $[e^{-z\Delta}\psi](0) = \langle (\text{ev } e^{-z\Phi\Delta\Phi^{-1}})^\sharp, \Phi\psi \rangle_{\text{Sym } V^*}$ by definition. As the scalar product $\langle \cdot, \cdot \rangle$ is non-degenerate on $\text{Sym } V^*$ and ψ can be chosen arbitrarily we conclude:

$$(\text{ev } e^{-z\Phi\Delta\Phi^{-1}})^\sharp = e^{z|x|^2} (2z)^N (e^{-z\Delta}(a(z)\Phi^{-1})). \quad (4)$$

Evidently the operators $e^{z|x|^2}$ and $e^{-z\Delta}$ are invertible with inverses $e^{-z|x|^2}$ and $e^{z\Delta}$ respectively, thus we can solve equation (4) uniquely for the infinite order jet of $a(z)$ leading to:

$$\begin{aligned} \text{jet}_0^\infty(a(z)\Phi^{-1}) &= e^{z\Delta} (2z)^{-N} e^{-z|x|^2} (\text{ev } e^{-z\Phi\Delta\Phi^{-1}})^\sharp \\ &= e^{z\Delta} (2z)^{-N} (\text{ev } e^{-z\Phi\Delta\Phi^{-1}} e^{z\Delta})^\sharp. \end{aligned} \quad (5)$$

Concerning the last step we remark that the polynomial $e^{-z|x|^2}$ corresponds to the differential operator $e^{z\Delta}$ with constant coefficients. Hence it is possible to replace $e^{-z|x|^2} (\text{ev } e^{-z\Phi\Delta\Phi^{-1}})^\sharp$ by $(\text{ev } e^{-z\Phi\Delta\Phi^{-1}} e^{z\Delta})^\sharp$, but not by $(\text{ev } e^{z\Delta} e^{-z\Phi\Delta\Phi^{-1}})^\sharp$, because the latter would involve partial derivatives of the coefficients of $e^{-z\Phi\Delta\Phi^{-1}}$ in the origin. Nevertheless we cheated a little bit, because while certainly injective the operator $(2z)^N$ is definitely not surjective. It is quite remarkable in itself that the value of the differential operator $e^{-z\Phi\Delta\Phi^{-1}} e^{z\Delta}$ in the origin lies in the image of $(2z)^N$ by equation (4), because its coefficients for the different powers z^r , $r \geq 0$, of z must have order less than or equal to r to have this true! Hence the inverse $(2z)^{-N}$ is well defined and refers to the unique preimage of $(\text{ev } e^{-z\Phi\Delta\Phi^{-1}} e^{z\Delta})^\sharp$ under $(2z)^N$.

Definition 2.6 (The Difference Operator of a Generalized Laplacian)

Consider a generalized Laplacian Δ and the flat Laplacian Δ acting on sections of \mathcal{E} and the trivial vector bundle $V \times E$ over V respectively. Associated to Δ and some smooth trivialization Φ of \mathcal{E} is the formal power series

$$\mathbb{I}(z) := e^{-z(\Phi\Delta\Phi^{-1})} e^{z\Delta} = 1 + z \mathbb{I}_1 + z^2 \mathbb{I}_2 + z^3 \mathbb{I}_3 + \dots$$

of differential operators on $V \times E$ called the difference operator. The values $\text{ev } \mathbb{I}_r$ of its coefficients have order less than or equal to r , so that $\text{ev } \mathbb{I}(z)$ lies in the image of $(2z)^N$.

At least in principle the recursion relation $\frac{d}{dz} \mathbb{I}(z) = \mathbb{I}(z)\Delta - (\Phi\Delta\Phi^{-1})\mathbb{I}(z)$ or equivalently $r\mathbb{I}_r = \mathbb{I}_{r-1}\Delta - (\Phi\Delta\Phi^{-1})\mathbb{I}_{r-1}$ for $r \geq 1$ together with $\mathbb{I}_0 = 1$ allows us to calculate $\mathbb{I}(z)$ to arbitrary order in z . However the composition $(\Phi\Delta\Phi^{-1})\mathbb{I}_{r-1}$ requires knowledge

of the full operator \mathbb{A}_{r-1} and not only of its value at the origin. Note that the recursion relation in itself does not imply that the values of the \mathbb{A}_r , $r \geq 0$, have order less than or equal to r , because the origin will be a singular point for the full operator $\mathbb{A}(z)$ in general. Reformulating equation (5) according to the definition of $\mathbb{A}(z)$ we arrive at the following theorem, which in a sense is the converse of Theorem 2.1:

Theorem 2.7 (Characterization of the Heat Kernel Coefficients)

The infinite order jet of the generating series $a(z)$ for the heat kernel coefficients of a generalized Laplacian Δ acting on sections of a vector bundle \mathcal{E} over V is uniquely characterized by its intertwining property $[e^{-z\Delta}\psi](0) = [e^{-z\Delta}(a(z)\psi)](0)$. In fact the infinite order jet of any solution $a(z)$ to this equation is given in terms of some smooth trivialization Φ of \mathcal{E} and the corresponding difference operator $\mathbb{A}(z)$ by:

$$\text{jet}_0^\infty(a(z)\Phi^{-1}) = e^{z\Delta} (2z)^{-N} (\text{ev } \mathbb{A}(z))^\sharp.$$

The essence of Theorem 2.7 is not particularly evident in the abstract formulation above and so we want to close this section expanding this formula for the jets of the heat kernel coefficients into powers of z . This exercise will show in particular that it is just as difficult to calculate the infinite order jets of the heat kernel coefficients a_k , $k \geq 0$, as it is to calculate the value of the powers Δ^r , $r \geq 0$, of Δ in 0 using normal coordinates and the chosen trivialization Φ of \mathcal{E} . In other words calculating the powers Δ^r explicitly is the problem of calculating the coefficients of the heat kernel expansion en nuce! Let us start by expanding the value $\text{ev } \mathbb{A}(z)$ into powers of z and homogeneous pieces

$$(\text{ev } \mathbb{A}(z))^\sharp = \sum_{r \geq s \geq 0} z^r \mathbb{A}_{r,s},$$

where each $\mathbb{A}_{r,s} \in \text{Sym}^s V^* \otimes \text{End } E$ is homogeneous polynomial of order s , and employ Theorem 2.7 to conclude

$$\text{jet}_0^\infty a(z) = \sum_{r \geq s \geq 2t \geq 0} \frac{z^t}{t!} \Delta^t \left(\frac{z^{r-s}}{2^s} \mathbb{A}_{r,s}^\sharp \right) = \sum_{k,l \geq 0} z^k \sum_{t=0}^k \frac{1}{2^{l+2t} t!} \Delta^t \mathbb{A}_{k+l+t, l+2t}^\sharp \quad (6)$$

by reordering the sum and setting $k := r - s + t$ and $l := s - 2t$. In particular the infinite order jet of the first coefficient a_0 picks up the principal symbols $\text{ev } \mathbb{A}_{l,l}$, $l \geq 0$, of the operators \mathbb{A}_l , $l \geq 0$, in the origin. Now by Definition 2.6 of the difference operator $\mathbb{A}(z)$ we find for the homogeneous pieces of its value

$$\mathbb{A}_{r,s}^\sharp = \sum_{\mu=0}^r (-1)^r \frac{1}{\mu!} |x|^{2\mu} \frac{1}{(r-\mu)!} (\text{ev } (\Phi \Delta \Phi^{-1})^{r-\mu})_{s-2\mu}^\sharp$$

where $(\text{ev } (\Phi \Delta \Phi^{-1})^{r-\mu})_{s-2\mu}^\sharp \in \text{Sym}^{s-2\mu} V^* \otimes \text{End } E$ is the homogeneous piece of order $s - 2\mu$ of the polynomial corresponding to $\text{ev } (\Phi \Delta \Phi^{-1})^{r-\mu}$. Inserting this expression into equation (6) above we get a closed formula for the homogeneous pieces of the infinite order jets of the heat kernel coefficients a_k , $k \geq 0$. Though explicit this formula looks rather awkward and

so we refrain from writing it down. In fact the main point is to understand the fantastic cancellation properties of the Δ_r^\sharp , $r \geq 0$, which make the $\text{ev } \mathbb{D}_r$, $r \geq 0$, operators of degree less than or equal to r in the first place! Without proper understanding of these cancellations we won't be able to take advantage of either formula to calculate the heat kernel coefficients explicitly.

References

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